Viscosity in Selfgravitating Accretion Disks

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ABSTRACT

We show that the standard model for geometrically thin accretion disks (α-disks) leads to inconsistencies if selfgravity plays a role. This problem arises from the parametrization of viscosity in terms of local sound velocity and vertical disk scale height. A viscosity prescription based on turbulent flows at the critical effective Reynolds number allows for consistent models of thin selfgravitating disks, and recovers the α-disk solution as the limiting case of negligible selfgravity. We suggest that such selfgravitating disks may explain the observed spectra of protoplanetary disks and yield a natural explanation for the radial motions inferred from the observed metallicity gradients in disk galaxies.

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1. Introduction

One of the major shortcomings of the current theoretical descriptions of accretion disks is lack of detailed knowledge about the underlying physics of viscosity in the disk. This problem is significant because almost all detailed modelling of the structure and evolution of accretion disks depends on the value of the viscosity and its dependence on the physical parameters. While there is general agreement that molecular viscosity is totally inadequate and that some kind of turbulent viscosity is required, there is far less certainty about its prescription. Most investigators adopt the so-called α-*ansatz* introduced by Shakura (1972) and Shakura and Sunyaev (1973) that gives the viscosity (ν) as the product of the pressure scale height in the disk (h), the velocity of sound (c_s), and a parameter α that contains all the unknown physics. One interprets this as some kind of isotropic turbulent viscosity $\nu = \nu_t = l_t v_t$ where l_t is an (unknown) length scale and v_t an (unknown) characteristic velocity of the turbulence. One may then write $\alpha = (v_t/c_s) \cdot (l_t/h)$. On general physical grounds neither term in parentheses can exceed unity so that $\alpha \leq 1$. If initially $v_t > c_s$, shock waves would result in strong damping and hence a return to a subsonic turbulent velocity. The condition $l_t > h$ would require anisotropic turbulence since the vertical length scales are limited by the disk's thickness, which is comparable to h.

A parameterization of this sort for ν is of course only of interest if the proportionality parameter, α, is (approximately) a constant. One can expect this to happen only if the scaling quantities are chosen in a physically appropriate manner. Models for the structure and evolution of accretion disks in close binary systems (e.g., dwarf novae and symbiotic stars) show that Shakura and Sunyaev's parameterization with a constant α leads to results that reproduce the overall observed behaviour of the disks quite well. As a result of this success, the α-ansatz is now used in practice in all sorts of accretion disks. It is noteworthy, however, that the α-ansatz retains no information about the mechanism generating the turbulence but only about limits to its efficiency.

In this contribution, we wish to

- show that simple application of the above viscosity prescription to the case of selfgravitating accretion disks leads to an apparent internal inconsistency (Sec. 2.2).

- propose a possible resolution of this problem in terms of a viscosity prescription (Sec. 3).

- discuss the implications of the new formulation of viscosity for the structure and evolution of accretion disks (Sec. 4).

- discuss protostellar and galactic disks as two examples where a non-standard viscosity description in selfgravitating disks is important (Sec. 5).

2. The Structure of Selfgravitating Accretion Disks

2.1. Conditions for Selfgravity in Accretion Disks

In the following we assume that the accretion disks are geometrically thin in the vertical direction and symmetric in the azimuthal direction. We approximate the vertical structure by a one zone model. Then a disk model is specified by the central mass M_*, the radial distributions of surface density $\Sigma(s)$, central plane temperature $T_c(s)$ and effective temperature $T_{\text{eff}}(s)$. The relevant material functions are the equation of state, the opacity and the viscosity prescription.

One can estimate the importance of selfgravity by comparing the respective contributions to the local gravitational accelerations in the vertical and radial directions.

The vertical gravitational acceleration at the disk surface is $2\pi G\Sigma$ and GM_*h/s^3, respectively, for the selfgravitating and the purely Keplerian case, respectively. Selfgravitation is thus dominant in the vertical direction when

$$\frac{M_d}{M_*} \sim \frac{\pi s^2 \Sigma}{M_*} > \frac{1}{2}\frac{h}{s}, \qquad (1)$$

where $M_d(s)$ is the enclosed mass in the disk and is given approximately by $M_d \sim \pi s^2 \Sigma$. With characteristic numbers for the relative geometrical thickness of non-selfgravitating disks of $h/s \sim 0.1 \ldots 0.02$, condition (1) translates into a condition for vertical selfgravity of $M_d/M_* > 0.01 \ldots 0.05$.

Similar considerations lead to the condition

$$M_d > M_* \qquad (2)$$

for selfgravitation to dominate in the radial direction. Thus, for increasing disk masses selfgravity first becomes important in the vertical direction.

2.2. Keplerian Selfgravitating Disks (KSG)

In this Section, we describe a selfgravitating (SG) accretion disk in the framework of the model by Shakura and Sunyaev (1973; hereafter referred to as the *standard model*), changing only the equation of hydrostatic equilibrium in the direction perpendicular to the disk's plane (*vertical* or z direction) to account for selfgravity in the direction perpendicular to the disk. Thus, while in the standard model the local vertical pressure gradient is balanced by the z component of the gravitational force due to the central object, in the SG case we have balance between two local forces, namely the pressure force and the gravitational force due to the disk's local mass. For the present purpose, we consider only selfgravity in the vertical direction, and assume that in the radial direction centrifugal forces are balanced by gravity from a central mass (*Keplerian* approximation). In the following, we will often refer to such accretion disks, in which selfgravity is important only in the vertical direction, as *Keplerian selfgravitating (KSG) disks*.

Hydrostatic equilibrium in the vertical direction yields

$$P = \pi G \Sigma^2 \qquad (3)$$

(Paczyński 1978), where P is the pressure in the central plane ($z = 0$), Σ is the surface mass density integrated in the z direction, and G is the gravitational constant. Equation (3) is strictly valid only in the limit of dominating selfgravity. We will comment on the case of marginal selfgravity later. To deduce Eq. (3), we have made a one-zone approximation for the disk's vertical structure, i.e., we have replaced derivatives with respect to the z direction by quotients of differences between the disk's surface and the central plane.

Since details of the thermodynamics in the z direction are of no particular relevance to our argument, we shall likewise assume the disk to be isothermal in the vertical direction.

Integrating the equation of conservation of angular momentum gives

$$\nu \Sigma = -\frac{\dot{M}}{s^3 \omega'} (s^2 \omega - \xi) \qquad (4)$$

with the radial mass flow rate[2] $\dot{M}$, the Keplerian rotational frequency ω, its radial derivative ω', and a quantity ξ allowing for the integration constant or, equivalently, for the inner boundary condition. For a detailed discussion of ξ see, e.g., Duschl & Tscharnuter (1991), Popham & Narayan (1995) and Donea & Biermann (1996). For simplicity, we set the boundary condition $\xi = 0$ in the subsequent discussion. This does not alter the essence of our argument, and only changes details close to the disk's inner radial boundary, since the product $s^2 \omega$ increases with s.

Finally, we use the standard viscosity *ansatz*

$$\nu = \alpha h c_{\mathrm{s}} \qquad (5)$$

and have for the sound velocity

$$c_{\mathrm{s}}^2 = P \left/ \left(\frac{\Sigma}{2h} \right) \right. . \qquad (6)$$

Equations (3) and (6) give $h = c_{\mathrm{s}}^2/(2\pi G\Sigma)$. Substituting into Eq. (5), we get $\nu\Sigma = (\alpha c_{\mathrm{s}}^3)/(2\pi G)$. Comparing this with Eq. (4), we find that the set of Eqs. (3) - (6) can be fulfilled only for a certain sound velocity, or temperature given by:

$$c_{\mathrm{s}}^3 = -\frac{2\pi G \dot{M}}{\alpha} \frac{\omega}{s\omega'} \sim -\frac{2\pi G \dot{M}}{\alpha} \left(\frac{\mathrm{d}\log s}{\mathrm{d}\log \omega} \right) . \qquad (7)$$

and hence

$$c_{\mathrm{s}}^2 = \frac{kT_{\mathrm{c}}}{m_H} = \left(\frac{2\pi}{3\alpha} G \dot{M} \right)^{2/3} . \qquad (8)$$

Thus, for a KSG disk, the α-ansatz leads to the requirement of a constant temperature for all radii s (or, if $\xi \neq 0$, the temperature is prescribed as a function of s), independent of thermodynamics. It also requires that the disk structure satisfy

$$h\Sigma = \frac{c_{\mathrm{s}}^2}{2\pi G} = \frac{\dot{M}^{2/3}}{(3\alpha)^{2/3} (2\pi G)^{1/3}} . \qquad (9)$$

In a standard non-SG accretion disk the temperature is a free parameter which is determined by the energy released by the inward flow of the disk gas ($\dot{M}$), by the local viscosity, and by the respective relevant cooling mechanisms. The viscosity depends on T_{c} via Eq.

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[2] We choose the convention of radial mass flow rate $\dot{M}$ and radial velocity v_s *positive* for inward motion.

(5) and on the equation of hydrostatic support in the direction normal to the disk, namely

$$\frac{h}{s} = \frac{c_{\mathrm{s}}}{v_\phi} \qquad (10)$$

($v_\phi = s\omega$) which in the non-SG case replaces Eq. (3).

In the KSG case, it is the surface density Σ (and hence h) which must adjust in order to radiate the energy deposited by viscous dissipation and provided by the inward flowing material. While detailed solutions are beyond the scope of this paper, they clearly exist formally. On the other hand, the normal *thermostat* mechanism does not operate, at least in the steady state. Indeed in certain circumstances, the condition of constant mid-plane temperature appears to be inconsistent with the basic thermodynamic requirement that the average gas temperature in the disk exceed that of the black body temperature required to radiate away the energy dissipated by viscous stresses (see Appendix). We are therefore doubtful whether a physically plausible and stable quasi-steady state solution exists.

Furthermore, the Jeans condition for fragmentation in the disk into condensations of radius R is

$$\frac{4\pi}{3} q G \overline{\rho} R^2 > c_{\mathrm{s}}^2 \qquad (11)$$

(see Mestel 1965) where q is factor of order unity. On the other hand, Eqs. (3) and (6) give

$$2\pi G \Sigma h = 4\pi G \overline{\rho} h^2 = c_{\mathrm{s}}^2 \qquad (12)$$

Thus, a selfgravitating disk is on the verge of fragmenting into condensations of radius $R \sim h$ unless these are destroyed by shear motion associated with the Keplerian velocity field.

Finally we note that the Reynolds number $\Re = vl/\nu_{\mathrm{mol}}$ in the disk flow is extremely high in any astrophysical context and this in itself is likely to lead to strong turbulence.

Indeed, as has been pointed out for example by Lynden-Bell and Pringle (1974) and by Thompson et al. (1977), the high Reynolds number character of the flow will lead to the generation of turbulence and hence to a steady enhancement in the effective (or Eddy) viscosity until the Reynolds number becomes subcritical. The limiting viscosity in this case is given for a disk as

$$\nu = \frac{1}{\Re_{\mathrm{crit}}} v_\phi s = \beta v_\phi s \qquad (13)$$

where $\Re_{\mathrm{crit}} \sim 1/\beta \sim 10^2 \ldots 10^3$. This form is quite different to the α-ansatz.

For all of the above reasons, we are led to consider alternative prescriptions for viscosity which may be more relevant to selfgravitating Keplerian (or fully selfgravitating) disks.

3. Prescription for Turbulent Viscosity

As noted in Sections 1 and 2.2 the need for some kind of turbulent viscosity in accretion disks is generally recognized, as is the very high Reynolds number of the flow in the absence of such a viscosity. In any event, it seems reasonable to assume that the turbulence is driven by the velocity field in the disk which has characteristic length and velocity scales s and v_ϕ, respectively. We might then expect that

$$\nu = v_{\mathrm{t}} l_{\mathrm{t}} \propto \Delta v_\phi \Delta s \qquad (14)$$

where Δv_ϕ and Δs are representative velocity and length scales of the flow. Furthermore, we may write

$$\Delta v_\phi = \frac{\partial v_\phi}{\partial s} \Delta s \sim \frac{v_\phi}{s} \Delta s \qquad (15)$$

If we consider turbulent elements in a *smoothed out* background gas with sound speed c_{s}, we may impose the limit that the turbulent velocities do not exceed c_{s}. Thus Eq. (15) gives

$$\frac{v_\phi}{s} \Delta s \sim \Delta v_\phi \leq c_{\mathrm{s}} \qquad (16)$$

or

$$\Delta s \sim \frac{s}{v_\phi} \Delta v_\phi \leq \frac{s}{v_\phi} c_{\mathrm{s}} \qquad (17)$$

From Eq. (10), this implies that for a standard Keplerian disk,

$$\Delta s \leq h \qquad (18)$$

and hence that

$$\nu \sim \alpha h c_{\mathrm{s}} \qquad (19)$$

where $\alpha \leq 1$. Note that the upper bound to Δs *implies* approximately isotropic turbulence. This is the standard α-ansatz but derived from considerations

of rotationally generated turbulence. The argument can, in fact, be inverted to show that under assumption of isotropic turbulence Eqs. (10) and (16), with $\Delta s = h$, imply $\Delta v_\phi \sim c_s$ which again yields the functional dependence of the α-ansatz. Implicit in this argument is that the effective viscosity is determined by the largest physically reasonable length and velocity scales.

For a SG disk (whether Keplerian or not) Eq. (10) is no longer valid so this analysis does not apply. In physical terms, the scale height in the disk no longer reflects global properties of the disk (mass of and distance to the central star) but is set by local conditions. Furthermore, if the material is significantly clumped, as seems likely, for example, in molecular clouds in galactic disks, an alternative prescription may be more appropriate. We suggest that in such situations (and possibly in others also) the appropriate length and velocity parameters may be the radial distance, s, and the azimuthal velocity, v_ϕ or, more specifically, some fractions of each, since in some sense they already represent the maximum of either scale available.

In support of this choice of s as the natural length scale we note that *(a)* it is the only length scale which is relevant for angular momentum transport which contains information about the driving agent for the turbulence – namely the rotation field, and *(b)* it is relevant to several other processes for transporting angular momentum such as spiral arms and large scale magnetic fields. Likewise, the orbital velocity is the only relevant velocity scale containing information on the rotation.

In these circumstances, we should obtain the *Reynolds* form for the viscosity, namely

$$\nu = \nu_{\mathrm{R}} = \beta s v_\phi \qquad (20)$$

where β is a constant. Since the rotation field is assumed to be driving the turbulence, it seems appropriate to restrict β using the condition that the effective Reynolds number of the flow not fall below the critical value $\Re_{\mathrm{crit}}$ for the onset of turbulence. Thus β should satisfy

$$\beta < \frac{1}{\Re_{\mathrm{crit}}} \sim 10^{-3} \ \ldots \ 10^{-2} \qquad (21)$$

In the following we will refer to Eq. (20) as the β-*ansatz* and to the disk structure arising from this viscosity prescription as β-*disks*. We suggest that

this formulation is most appropriate for selfgravitating disks which we explore further in Sec. 4.

We note, however, that if the disk matter distribution is clumpy (e.g., clouds within a low density smoothed out distribution) then there is a formal connection between the α- and the β-prescriptions. Since in the β-formulation the clump velocities are of order v_ϕ shock heating will tend to heat the low density interclump gas until its sound speed $c_s \sim v_\phi$. The interclump gas will then have a scale height $h \sim s$, the scale of the clumpy disk, and will hence be roughly a spherical structure. At this point the α- and β-prescriptions look formally identical but the scale height and sound speed now refer to a more or less spherical background distribution of hot gas in which a disk structure of cloudy clumps is inbedded.

4. Structure of Selfgravitating β Disks

For SG β-disks, Eqs. (3), (4) and (6) are applicable, but Eq. (5) must be replaced by Eq. (20). It then follows that

$$\Sigma = -\frac{\dot{M}}{\beta s^3 \omega'} \qquad (22)$$

and that

$$c_s^2 = -\frac{\pi G h}{\nu} \frac{\dot{M}}{s\omega'} = -\frac{\pi G h \dot{M}}{\beta s^3 \omega'} \qquad (23)$$

Thus the SG β-disks recover the thermostat property of the standard disk, namely that the temperature and scale height can adjust to accomodate (radiate away) the energy input to the system from viscous dissipation and inward motion.

4.1. Keplerian Selfgravitating Disks

For the particular case of a KSG β-disk we have:

$$\frac{c_s^2}{h} = \frac{4}{3} \frac{\pi G \dot{M}}{\beta} \frac{1}{(GM_*)^{1/2} s^{1/2}} \qquad (24)$$

For the SG β-disk it follows immediately from Eq. (22) and from mass conservation in the disk that the radial inflow velocity v_s is given by

$$v_s = \frac{\dot{M}}{2\pi s \Sigma} = -\frac{\beta s^2 \omega'}{2\pi} \qquad (25)$$

For the KSG β-disk, we then have

$$v_s = \frac{3}{4\pi}\beta s\omega = \frac{3}{4\pi}\beta v_\phi \qquad (26)$$

Thus at each radius the inward velocity is the same fraction of the local orbital velocity. From Eq. (25) this, in fact, holds for any SG β-disk in which the angular velocity is a power law function of s with adjustment only to the numerical factor in Eq. (26). If β satisfies the constraint (21), then the approximation of centrifugal balance in the radial direction remains well justified.

Under these conditions the dissipation per unit area of a SG β-disk is given by

$$D = \frac{\dot{M}}{4\pi s}\left(\frac{v_\phi^2}{s}\right) = 2\sigma T_{\text{eff}}^4 \qquad (27)$$

where σ is the Stefan-Boltzmann constant. For an optically thick KSG disk this yields the same radial dependence of T_{eff} as for the standard disk, namely

$$T_{\text{eff}} = \left(\frac{G\dot{M}M_*}{8\pi\sigma}\right)^{1/4} s^{-3/4} \qquad (28)$$

This temperature dependence which is identical to that of the standard model then leads to the well known energy distribution for an optically thick standard disk of $S_\nu \propto \nu^{1/3}$. This also implies that – as long as the disks are not fully self-gravitating – it is hard to distinguish between an α- and a β-disk model observationally.

4.2. Fully Selfgravitating Disks (FSG)

We turn now to the case of the fully selfgravitating (FSG) β-disk, in which the disk mass is sufficiently great that it dominates the gravitational terms in the hydrostatic support equation in both the radial and vertical directions. While there are many potential solutions for the FSG disk structure, one is well known in both mathematical and observational terms, namely the constant velocity (v_ϕ = const.) disk. Within such a disk structure we have simultaneous solutions to the equation of radial hydrostatic equilibrium and Poisson's equation of the form

$$v_\phi = s\omega = v_0 \quad \text{and} \quad \Sigma \propto \Sigma_0\left(\frac{s}{s_0}\right)^{-1}. \qquad (29)$$

(Toomre 1963; Mestel 1963). For the FSG disk, Eq. (22) then leads to

$$\Sigma = \frac{\dot{M}}{\beta s v_0} \qquad (30)$$

which has the same radial dependence as the structural solution as shown in Eq. (29). Thus Eq. (30) may be viewed as giving the rate of mass flow through the disk for a FSG β-disk with constant rotational velocity v_0. Finally, the equation of continuity provides a constraint if the structure is to maintain a basically steady state structure. We then have:

$$s\frac{\partial\Sigma}{\partial t} = \frac{\partial}{\partial s}(sv_s\Sigma) = \frac{3}{4\pi}\frac{\partial}{\partial s}(\beta v_\phi s\Sigma) \qquad (31)$$

For the $v_\phi = v_0$ = const. disk, it then follows from Eq. (30) that $\partial\Sigma/\partial t = 0$. Thus the constant velocity disk represents a steady state solution in regions sufficiently far from the inner and outer boundaries of the β-disk.

It is then possible, in the spirit of the discussion of Eqs. (27) and (28), to calculate the energy dissipation rate per unit area D for the constant velocity β-disk. We then find

$$D = 2\sigma T_{\text{eff}}^4 = \frac{\dot{M}v_0^2}{4\pi s^2} \qquad (32)$$

so that

$$T_{\text{eff}} = \left(\frac{\dot{M}v_0^2}{8\pi\sigma}\right)^{1/4} s^{-1/2} \qquad (33)$$

For flux density S_ν emitted by an optically thick, constant velocity β-disk it then follows that

$$S_\nu \propto \nu^{-1} \qquad (34)$$

In reality, a sufficiently massive disk may be expected to have an inner Keplerian (standard) zone, a Keplerian selfgravitating zone (KSG), and a fully selfgravitating zone (FSG). We should therefore expect a smooth transition in the spectral energy distribution from the $\nu^{1/3}$ spectrum of the inner two zones to the ν^{-1} spectrum arising at longer wavelengths from the FSG zone. One could turn this argument around and argue that, if no other components contribute to the spectrum, the flatness of the νF_ν distribution is a measure for the importance of selfgravity and thus for the relative mass of the accretion disk as compared to the central accreting object. This, of course, applies only to the optically thick case which may not arise frequently in strongly clumped disks.

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4.3. Time Scales

The evolution of accretion disks can be described by a set of time scales. For our purposes, the dynamical and the viscous time scale are of particular interest.

The dynamical time scale τ_{dyn} is given by

$$\tau_{\mathrm{dyn}} = \frac{1}{\omega}. \tag{35}$$

While this formulation applies to all cases, selfgravitating or not, it is only in the non-SG and in the KSG cases that ω is given by the mass of the central accretor and by the radius. In the FSG case, ω is determined by solving Poisson's equation.

The timescale of viscous evolution τ_{visc} is given by

$$\tau_{\mathrm{visc}} = \frac{s^2}{\nu} \tag{36}$$

In the standard non-SG and geometrically thin ($h \ll s$) case, this leads to

$$\tau_{\mathrm{visc}}^{\mathrm{non-SG}} = \left(\frac{s}{h}\right)^2 \frac{\tau_{\mathrm{dyn}}}{\alpha} \gg \frac{\tau_{\mathrm{dyn}}}{\alpha}. \tag{37}$$

In KSG and FSG disks, τ_{visc} is given by

$$\tau_{\mathrm{visc}}^{\mathrm{KSG}} = \tau_{\mathrm{visc}}^{\mathrm{FSG}} = \tau_{\mathrm{visc}}^{\mathrm{SG}} = \frac{\tau_{\mathrm{dyn}}}{\beta} \tag{38}$$

With $\alpha < 1$ and $\beta \ll 1$ (Eq. 21) under all circumstances $\tau_{\mathrm{visc}} \gg \tau_{\mathrm{dyn}}$. In the SG cases the ratio between the two timescales decouples from the disk structure. In all cases the models are self consistent in assuming basic hydrostatic equilibrium in the vertical direction.

5. Possible Applications

5.1. Protoplanetary Accretion Disks

T Tauri stars have infrared spectral energy distributions νF_ν which can be approximated in many cases by power laws $\nu F_\nu \propto \nu^n$ with a spectral index n in the range $\sim 0 \ldots 1.3$. Assuming this spectral behaviour to be due to radiation from an optically thick disk, it translates into a radial temperature distribution $T_{\mathrm{eff}} \propto s^{-q}$ with $n = 4 - 2/q$.

A *passive* disk that only reradiates reprocessed radiation from the central star has $q = 3/4$ or $n = 4/3$. An optically thick *active* non-selfgravtating accretion disk which radiates energy that is liberated through viscous dissipation shows the same spectral distribution. Thus, if accretion disks were the only contributors to T Tauri spectra, and if the disks were optically thick and non-selfgravitating, the spectral shape should be the same for all objects. This is in clear contrast to the above mentioned observed spectral energy distributions.

Adams, Lada and Shu (1988) were the first to discuss the possibility of a non-standard radial temperature distribution with $q \neq 3/4$. Using q as a free parameter, they find that for flat spectrum sources, their best fits require disk masses that are no longer very small as compared to the accreting stars' masses. They already mention the possibility that the flatness of the spectrum and selfgravity of the disk may be related. On the other hand, at that time this indirect argument was the only evidence for large disk masses.

Due to the lack of accretion disk models yielding the appropriate values of q, since the work of Adams, Lada and Shu several other interpretations for the flatness of the spectra were offered, for instance dusty envelopes engulfing a star with a standard disk around it (Natta 1993).

In the meantime, high resolution direct observations of protostellar disks yield independent strong evidence for comparatively large disk masses. Lay et al. (1994), for instance, find a lower limit for the disk masses in HL Tau – one of the sources in Adams, Lada and Shu sample of flat spectrum T Tauri stars – of $\sim 0.02 \, \mathrm{M_\odot}$.

We suggest that the flatness of the spectrum actually reflects the mass of the disk, i.e., the importance of selfgravity. For disk masses considerably smaller than $\sim 1/30 M_*$, the standard accretion disk models apply. For disks whose masses are larger but still small compared to M_* the spectral behaviour is not altered significantly. but disk structure and the time scale of disk evolution (τ_{visc}, see Eqs. 37 and 38) change. For even more massive disks, we expect a clear trend towards flatter spectra that approach an almost constant νF_ν distribution if selfgravity in the disk becomes important in radial as well as in vertical direction. But the time scale for the disk evolution remains the same (Eq. 38).

5.2. Galactic Disks

The relevance of viscosity in the evolution of galactic disks has been the subject of discussion since von Weizsäcker (1943, 1951) and Lüst (1952) first raised

the issue nearly fifty years ago. They noted then that, with an eddy viscosity formulation (a β-disk), the time scale for evolution of typical galactic disks was comparable to the age of the universe and suggested that this might account for the difference between spiral and elliptical galaxies.

With the subsequent realization that galactic disks moved primarily under the influence of extended massive halos, interest in FSG disks waned. However, as noted above, it is possible for a massive disk to exist and evolve under the influence of viscosity embedded in such a halo gravitational field. Indeed, in the event that such a structure forms, it must evolve under viscous dissipation and can achieve a quasi-steady state with essentially the same mass and energy dissipation distribution as for the FSG constant velocity disk. We refer to this case as an Embedded Self Gravitating (ESG) disk.

The timescale for viscous evolution τ_{visc} as given in section 4.3 suggests a means of differentiating between the α- and β-formulations for this case. For a normal spiral galaxy with a suggested mean temperature in the gaseous disk of around 10^4 K and a scale height of around 300 pc, we obtain

$$
\begin{aligned}
\tau_{\mathrm{visc}}^{(\alpha)} &\sim 10^4 \tau_{\mathrm{dyn}} \sim 3\,10^{11}\ \mathrm{yr} \\
\tau_{\mathrm{visc}}^{(\beta)} &\sim 10^2 - 10^3 \tau_{\mathrm{dyn}} \sim 3\,10^9 - 3\,10^{10}\ \mathrm{yr}
\end{aligned} \tag{39}
$$

Thus, with these parameters, little evolution would take place in a Hubble time on the α-hypothesis but significant evolution is predicted on the β-hypothesis.

In terms of inflow velocities the β-ansatz suggests values in the range $0.3 - 3\,\mathrm{km\,s}^{-1}$ which would be exceedingly hard to measure directly: the α-ansatz suggests still lower values. On the other hand, it may be possible to provide limits on the viscosity through other observational constraints. For example, the build up of the 3 kpc molecular ring in our own galaxy can be interpreted as due to viscosity driven inflow in the constant velocity part of the galactic disk which ceases (or at least slows down) in the constant angular velocity inner regions (Icke 1979; Däther and Biermann 1990). Similarly, several authors have suggested that the radial abundance gradients observed in our own and other disk galaxies may be due to radial motion and diffusive mixing associated with the turbulence generating the eddy viscosity (Lacey & Fall 1985; Sommer-Larsen and Yoshii 1990; Köppen 1994; Edmunds & Greenhow 1995, Tsujimoto, Yoshii,

Nomoto and Shigeyama 1995). According to these authors, radial inflows of around $1\,\mathrm{km\,s}^{-1}$ at the galactic location of the Sun are required for optimum fits to the abundance gradient data within the context of the viscous disk hypothesis. Such inflow velocities are consistent with the β-ansatz but could, of course, be generated also by other means (e.g effects of bars, magnetic fields).

6. Summary

The standard model for geometrically thin accretion disks with viscosity proportional to sound velocity and vertical scale height (often referred to as α-disks) leads to inconsistencies if the disk's mass is large enough for selfgravity to play a role. This problem arises even in Keplerian selfgravitating disks in which only the vertical structure is dominated by selfgravity while the azimuthal motion remains Keplerian.

We propose a viscosity prescription based on the assumption that the effective Reynolds number of the turbulence does not fall below the critical Reynolds number. In this parametrization the viscosity is proportional to the azimuthal velocity and the radius (β-disks). This prescription yields physically consistent models of both Keplerian and fully selfgravitating accretion disks. Moreover, for the case of sufficiently small disk masses, we recover the α-disk solution as a limiting case.

Such β-disk models may be relevant to protoplanetary accretion disks as well as for galactic disks. In the case of protoplanetary disks they yield spectra that are considerably flatter than those due to nonselfgravitating disks, in better agreement with observed spectra of these objects. In galactic disks, they result in viscous evolution on time scales shorter than the Hubble time and thus offer a natural explanation for an inward flow that could account for the observed chemical abundance gradients.

A. Thermodynamic Considerations for KSG α-disks

For a KSG α-disk, we have from Eq. 8 that

$$T_{\mathrm{c}} = \left(\frac{2\pi}{3\alpha}GM\dot{M}\right)^{2/3}\frac{m_{\mathrm{H}}}{k} = 2.41\,10^5\,\mathrm{K}\,\left(\frac{\dot{m}}{\alpha_{-1}}\right)^{2/3} \tag{A1}$$

where $\dot{m}$ is the mass flow rate in solar masses per year, and $\alpha_{-1} = \alpha/0.1$.

If the disk is optically thick and advection is negligible, viscous dissipation leads to local effective temperature of

$$\begin{aligned} T_{\mathrm{eff}} &= \left(\frac{3}{8\pi\sigma}\right)(GM)^{1/4}\dot{M}^{1/4}s^{-3/4} \\ &= 8.53\,10^3\,\mathrm{K}\,\left(\frac{m\dot{m}}{s_{\mathrm{A}}^3}\right)^{1/4} \end{aligned} \tag{A2}$$

with m the mass of the central star in solar units and s_{A} the radius in astronomical units.

An essential thermodynamics requirement is that $T_{\mathrm{c}} > T_{\mathrm{eff}}$ or that

$$\frac{T_{\mathrm{eff}}}{T_{\mathrm{c}}} = 3.53\,10^{-2}\frac{m^{1/4}\dot{m}^{-5/12}}{s_{\mathrm{A}}^{3/4}}\alpha_{-1}^{2/3} < 1 \tag{A3}$$

This condition is satisfied provided that

$$\dot{m} > \dot{m}_{\mathrm{T}} = 3.27\,10^{-4}\frac{m^{3/5}\alpha_{-1}^{8/5}}{s_{\mathrm{A}}^{9/5}} \tag{A4}$$

and that the disk is selfgravitating in the vertical direction at s_{A}. The latter condition leads to a second requirement on $\dot{m}$.

For a standard Keplerian disk, the mass flow rate is given (Eqs. 10,19,36) by

$$\dot{M} \approx \frac{M_{\mathrm{d}}}{\tau_{\mathrm{visc}}} \approx \frac{M_{\mathrm{d}}\nu}{s^2} = \alpha M_{\mathrm{d}}\left(\frac{h}{s}\right)^2\omega \tag{A5}$$

with M_{d} the disk's mass. From Eq. 1, the condition that the disk is non-selfgravitating is $M_{\mathrm{d}} < (h/2s)M_*$ and hence, from Eq. A5, that

$$\dot{M} < \frac{\alpha}{2}\left(\frac{h}{s}\right)^3\left(\frac{GM_*^3}{s^3}\right)^{1/2} \tag{A6}$$

or

$$\dot{m} < \dot{m}_{\mathrm{G}} = 3.14\,10^{-1}\alpha_{-1}\left(\frac{h}{s}\right)^3\frac{m^{3/2}}{s_{\mathrm{A}}^{3/2}} \tag{A7}$$

A selfconsistent and physically acceptable solution can be obtained only if $\dot{m}_{\mathrm{G}} > \dot{m}_{\mathrm{T}}$, that is the disk becomes selfgravitating at values of $\dot{m}$ which are sufficiently high that thermodynamic requirements are not violated. This condition may then be written as

$$\frac{h}{s} > 1.01\,10^{-1}m^{-3/10}s_{\mathrm{A}}^{-1/10}\alpha_{-1}^{1/5} \tag{A8}$$

Thus *thin* KSG α-disks with $m \ggg 1$, $s_{\mathrm{A}} \ggg 1$ appear to be inconsistent with basic thermodynamic requirements.

No such inconsistencies occur for β-disks for which the normal thermostat effect is free to operate.

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